Contemporary Mathematics

The coset leader and list weight enumerator

Relinde Jurrius and Ruud Pellikaan

In Topics in Finite Fields
11th International Conference on Finite Fields and their Applications
G.M. Kyureghyan, G.L. Mullen & A. Pott (Eds.)
AMS, Contemporary Mathematics Series, vol. 632, pp. 229-252, 2015
Corrected version, 7 November 2019

Abstract. This paper is about the coset leader and list weight enumerators and their extensions using the theory of arrangements of hyperplanes, geometric lattices and characteristic and Möbius polynomials.

1. Introduction

The probability of error in error-detection can be expressed in terms of the weight enumerator of a code [**23**], and for error-correction the coset leader weight enumerator is used [**24**]. The coset leader weight enumerator is also used in steganography to compute the average of changed symbols [**25, 26**]. The computation of the weight enumerator of a code is NP-hard [**3, 4, 31**]. The complexity of computing the coset leader weight enumerator of a code is considered extremely difficult [**15**]. The size of lists of nearest codewords is considered in the list decoding of Reed-Solomon codes [**21, 30**]. This motivates the definition of the list weight enumerator and its extension. This research originated in [**19**].

Let C be a linear code of length n over $\mathbb{F}_q$ and let $\mathbf{y} \in \mathbb{F}_q^n$. The weight of the coset $\mathbf{y}+C$ is defined by $\mathrm{wt}(\mathbf{y}+C) = \min\{\mathrm{wt}(\mathbf{y}+\mathbf{c}) : \mathbf{c} \in C\}$. A *coset leader* is a choice of an element $\mathbf{y} \in \mathbb{F}_q^n$ of minimal weight in its coset, that is $\mathrm{wt}(\mathbf{y}) = \mathrm{wt}(\mathbf{y}+C)$. Let α_i be the number of cosets of C that are of weight i. Let λ_i be the number of vectors $\mathbf{y}$ in $\mathbb{F}_q^n$ that are of minimal weight i in its coset. Then we can define the *coset leader* and *list weight enumerator*:

$$\alpha_C(X,Y) = \sum_{i=0}^{n} \alpha_i X^{n-i}Y^i \quad \text{and} \quad \lambda_C(X,Y) = \sum_{i=0}^{n} \lambda_i X^{n-i}Y^i.$$

2010 *Mathematics Subject Classification.* 94B05.
Relinde Jurrius is supported by VUB-grant GOA62.

©0000 (copyright holder)

There is a one-to-one correspondence between cosets and syndromes. It is a well known fact that a coset leader corresponds to a minimal way to write its syndrome as a linear combination of the columns of a parity check matrix. This idea gives rise to the *derived code* code $D(C)$ and its associated *arrangement of hyperplanes*. An inclusion/exclusion counting argument in the derived code gives us a way to determine the (extended) coset leader and (extended) list weight enumerator. A similar argument using the code itself gives a way to determine the weight enumerator, see [**17**] and [**20**]. We consider several examples, including the Hamming codes and some generalized Reed-Solomon codes.

We address the questions whether the *extended* coset leader weight enumerator and/or *extended* list weight enumerator of C determine the corresponding enumerators of $C^\perp$, as is the case for the ordinary weight enumerator by the MacWilliams relations. This problem has a negative answer for the ordinary coset leader weight enumerator by [**2**]. See [**24**, Research Problem 5.1].

2. Codes, arrangements and geometric lattices

For an overview about the relation between codes, projective systems, hyperplane arrangements, and geometric lattices, see [**18, 17, 20**]. In this section we briefly recapture some necessary theory.

Let G be the generator matrix of a linear $[n,k]$ code C over $\mathbb{F}_q$. Then we can form the $[n,k]$ code $C\otimes\mathbb{F}_{q^m}$ over $\mathbb{F}_{q^m}$ by taking all $\mathbb{F}_{q^m}$-linear combinations of the codewords in C. We call this the *extension code* of C over $\mathbb{F}_{q^m}$. We can determine the weight enumerator of such an extension code by using only the code C. By embedding its entries in $\mathbb{F}_{q^m}$, we find that G is also a generator matrix for the extension code $C\otimes\mathbb{F}_{q^m}$.

Definition 2.1. The *extended weight enumerator* $W_C(X,Y,T)$ of a linear code of length n is a homogeneous polynomial in X and Y of degree n with coefficients $A_w(T)$:

$$W_C(X,Y,T)=\sum_{w=0}^{n}A_w(T)X^{n-w}Y^w.$$

The $A_w(T)$ are integral polynomials in T such that $A_w(q^m)$ is the number of codewords of weight w in $C\otimes\mathbb{F}_{q^m}$.

Let $\mathbb{F}$ be a field. A *projective system* $\mathcal{P}=(P_1,\ldots,P_n)$ in $\mathbb{P}^r(\mathbb{F})$, the projective space over $\mathbb{F}$ of dimension r, is an n-tuple of points P_j in this projective space, such that not all these points lie in a hyperplane.
An n-tuple $(H_1,\ldots,H_n)$ of hyperplanes in $\mathbb{F}^k$ is called an *arrangement* in $\mathbb{F}^k$. The arrangement is called *simple* if all the n hyperplanes are mutually distinct. The arrangement is called *central* if all the hyperplanes are linear subspaces. A central arrangement is called *essential* if the intersection of all its hyperplanes is equal to $\{0\}$.

Let L be a poset (partially ordered set) and let $x,y\in L$. Then x and y have a *least upper bound* if there is a $z\in L$ such that $x\leq z$ and $y\leq z$, and if $x\leq w$ and $y\leq w$, then $z\leq w$ for all $w\in L$. If x and y have a least upper bound, then such an element is unique and it is called the *join* of x and y and denoted by $x\vee y$. Similarly

the *greatest lower bound* of x and y is defined. If it exists, then it is unique and it is called the *meet* of x and y and denoted by $x \wedge y$. A poset L is called a *lattice* if $x \vee y$ and $x \wedge y$ exist for all $x, y \in L$.

Let L be a lattice with minimum 0. An *atom* is an element $a \in L$ that is a cover of 0. A lattice is called *atomic* if for every $x > 0$ in L there exist atoms $a_1, \ldots, a_r$ such that $x = a_1 \vee \cdots \vee a_r$. The minimum length of a chain from 0 to x is called the *rank* of x and is denoted by $r_L(x)$ or $r(x)$ for short. A lattice is called *semimodular* if it has a rank function that satisfies $r(x \vee y) + r(x \wedge y) \leq r(x) + r(y)$ for all $x, y \in L$. A lattice L is called a *geometric lattice* if it is atomic and semimodular and has no infinite chains. If L is a geometric lattice L, then it has a minimum and a maximum and $r(1)$ is called the rank of L and is denoted by $r(L)$.

Via the columns of a generator matrix, we can associate a projective system – or, dually, an arrangement of hyperplanes – to a linear code. In the next examples we see that these structures give rise to a geometric lattice. These are Examples 5.45 and 5.46 from [**20**].

EXAMPLE 2.2. Let $\mathbb{F}$ be a field and let $\mathcal{V} = (\mathbf{v}_1, \ldots, \mathbf{v}_n)$ be an n-tuple of nonzero vectors in $\mathbb{F}^k$, i.e., a projective system in $\mathbf{P}^{k-1}(\mathbb{F})$. Let $L = L(\mathcal{V})$ be the collection of all linear subspaces of $\mathbb{F}^k$ that are generated by subsets of $\mathcal{V}$, with inclusion as partial order. This poset is a lattice, because for all $x, y \in L$ we can determine their join and meet by $x \vee y = x + y$ and

$$x \wedge y = \bigvee \{z : z \leq x, z \leq y\}.$$

The minimum of L is $0_L = \{\mathbf{0}\}$, the linear subspace generated by the empty set, and the maximum 1_L is given by is the subspace generated by all $\mathbf{v}_1, \ldots, \mathbf{v}_n$.
To show that the lattice L is geometric, we have to show that it is atomic, semimodular and without infinite chains. The latter is obvious since L is finite. Let a_j be the linear subspace generated by $\mathbf{v}_j$. Then $a_1, \ldots, a_n$ are the atoms of L. Let x be the subspace generated by $\{\mathbf{v}_j : j \in J\}$. Then $x = \bigvee_{j \in J} a_j$. If x has dimension r, then there exists a subset I of J such that $|I| = r$ and $x = \bigvee_{i \in I} a_i$. Hence L is atomic and $r(x) = \dim(x)$. Now $x \wedge y \subseteq x \cap y$, so

$$r(x \vee y) + r(x \wedge y) \leq \dim(x + y) + \dim(x \cap y) = r(x) + r(y).$$

Hence the semimodular inequality holds. We conclude that L is a geometric lattice.

EXAMPLE 2.3. Let $\mathbb{F}$ be a field and let $\mathcal{A} = (A_1, \ldots, A_n)$ be an arrangement over $\mathbb{F}$ of hyperplanes in the vector space $V = \mathbb{F}^k$. Let $L = L(\mathcal{A})$ be the collection of all nonempty intersections of elements of $\mathcal{A}$. By definition $\mathbb{F}^k$ is the empty intersection. Define the partial order $\leq$ by

$$x \leq y \quad \text{if and only if} \quad y \subseteq x.$$

Then V is the minimum element. Furthermore

$$x \vee y = x \cap y \ \text{ if } \ x \cap y \neq \emptyset, \quad \text{and} \quad x \wedge y = \bigcap \{z : x \cup y \subseteq z\}.$$

Suppose that $\mathcal{A}$ is a central arrangement. Then the intersection of all hyperplanes is not empty and the maximum element. If this intersection is equal to $\{0\}$, then the arrangement is called essential. Moreover $x \cap y$ is nonempty for all $x, y \in L$, so $x \vee y$ and $x \wedge y$ exist for all $x, y \in L$ and L is a lattice.

We will show that L is atomic by showing it is isomorphic to the lattice in the previous example. Let $\mathbf{v}_j = (v_{1j}, \ldots, v_{kj})$ be a nonzero vector such that $\sum_{i=1}^{k} v_{ij} X_i = 0$ is a homogeneous equation of A_j. Let $\mathcal{V} = (\mathbf{v}_1, \ldots, \mathbf{v}_n)$ and let x be the subspace generated by $\{\mathbf{v}_j : j \in J\}$. Consider the map $\varphi : L(\mathcal{V}) \to L(\mathcal{A})$ defined by

$$\varphi(x) = \bigcap_{j \in J} A_j.$$

Now $x \subset y$ if and only if $\varphi(y) \subset \varphi(x)$ for all $x, y \in L(\mathcal{V})$. Therefore, φ is a strictly monotone map. Furthermore φ is a bijection and its inverse map is also strictly monotone. Hence $L(\mathcal{V})$ and $L(\mathcal{A})$ are isomorphic lattices. Therefore $L(\mathcal{A})$ is also a geometric lattice.

We will now define some important polynomials associated to geometric lattices.

DEFINITION 2.4. Let L be a finite geometric lattice with Möbius function $\mu(x, y)$. The *characteristic polynomial* of L is defined by

$$\chi_L(T) = \sum_{x \in L} \mu(0, x) T^{r(L) - r(x)}.$$

The *two-variable characteristic polynomial* or *coboundary polynomial* is defined by

$$\chi_L(S, T) = \sum_{x \in L} \sum_{x \leq y \in L} \mu(x, y) S^{a(x)} T^{r(L) - r(y)}$$

where $a(x)$ is the number of atoms a in L such that $a \leq x$.

Note that $\mu(0, 1) = \chi_L(0)$ and $\chi_L(0, T) = \chi_L(T)$, because for $S = 0$ the only nonzero term has $a(x) = 0$, so $x = 0_L$.

REMARK 2.5. Let n be the number of atoms of L. Then the following relation holds for the coboundary polynomial in terms of characteristic polynomials:

$$\chi_L(S, T) = \sum_{i=0}^{n} S^i \chi_i(T) \quad \text{with} \quad \chi_i(T) = \sum_{\substack{x \in L \\ a(x) = i}} \chi_{L_x}(T).$$

Here L_x is the geometric lattice of all elements bigger than or equal to $x \in L$. The polynomial $\chi_i(T)$ is called the *i-defect* polynomial. See [**9, 8**].

DEFINITION 2.6. Let L be a finite geometric lattice. The *two variable Möbius polynomial* $\mu_L(S, T)$ is defined by

$$\mu_L(S, T) = \sum_{x \in L} \sum_{x \leq y \in L} \mu(x, y) S^{r(x)} T^{r(L) - r(y)}.$$

Note that $\mu_L(0, T) = \chi_L(0, T) = \chi_L(T)$.

REMARK 2.7. Let r be the rank of the geometric lattice L. Then the following relation holds for the Möbius polynomial in terms of characteristic polynomials:

$$\mu_L(S, T) = \sum_{i=0}^{r} S^i \mu_i(T) \quad \text{with} \quad \mu_i(T) = \sum_{x \in L_i} \chi_{L_x}(T).$$

The Möbius polynomial was introduced by Zaslavsky [**32**, Section 1] for hyperplane arrangements and for signed graph colorings in [**33**, Section 2] where it is called the Whitney polynomial. See also [**1**] and [**20**, §5.8.2].

We give some more background on the determination of the coboundary and Möbius polynomial of an arrangement. As we will see later, the method for determining the coset leader weight enumerator has many similarities with the following.

PROPOSITION 2.8. *Let q be a prime power, and let $\mathcal{A} = (A_1, \ldots, A_n)$ be a simple and essential arrangement in $\mathbb{F}_q^k$. Then*

$$\chi_{\mathcal{A}}(q^m) = |\mathbb{F}_{q^m}^k \setminus (A_1 \cup \cdots \cup A_n)|.$$

PROOF. See [**1**, Theorem 2.2] and [**5**, Proposition 3.2] and [**11**, Sect. 16] and [**27**, Theorem 2.69] and [**20**, Proposition 5.45]. □

REMARK 2.9. Let $\mathcal{A} = (A_1, \ldots, A_n)$ be a simple and essential arrangement in $\mathbb{F}^k$. Let x be an element of $L = L(\mathcal{A})$, i.e., an intersection of hyperplanes, with dimension l. The *restriction* $\mathcal{A}_x$ is the arrangement in $\mathbb{F}^l$ of all hyperplanes $x \cap A_j$ such that $x \cap A_j \neq \emptyset$ and $x \cup A_j \neq x$. Then $L(\mathcal{A}_x) = L_x$. Let $\cup\mathcal{A}_x$ be the union of the hyperplanes of $\mathcal{A}_x$. Then by Proposition 2.8, we have

$$\chi_{L_x}(q^m) = |(x \setminus (\cup\mathcal{A}_x))(\mathbb{F}_{q^m})| \,.$$

So $\chi_{L_x}(q^m)$ counts the number of vectors that are in $x \subseteq \mathbb{F}^k$ but not in any element of L bigger than x. Or, equivalently, it counts the number of vectors that are in x but not in any other element of L_x.

DEFINITION 2.10. From now on $\{1, \ldots, n\}$ will be abbreviated by $[n]$. Let $\mathcal{A} = (A_1, \ldots, A_n)$ be an essential simple arrangement over the field $\mathbb{F}$ in $\mathbb{F}^k$ and let $J \subseteq [n]$. Define $A_J = \cap_{j \in J} A_j$. Then A_J are elements of a geometric lattice with rank function r as in Example 2.3. Consider the decreasing sequence

$$\mathcal{N}_k \subset \mathcal{N}_{k-1} \subset \cdots \subset \mathcal{N}_1 \subset \mathcal{N}_0$$

of algebraic subsets of the affine space $\mathbb{A}^k$, defined by

$$\mathcal{N}_i = \bigcup_{\substack{J \subseteq [n] \\ r(A_J) = i}} A_J.$$

Define $\mathcal{M}_i = (\mathcal{N}_i \setminus \mathcal{N}_{i+1})$.

Note that $\mathcal{N}_0 = \mathbb{A}^k$, $\mathcal{N}_1 = \cup_{j=1}^n A_j$, $\mathcal{N}_k = \{0\}$ and $\mathcal{N}_{k+1} = \emptyset$. Furthermore, $\mathcal{N}_i$ is a union of linear subspaces of $\mathbb{A}^k$ all of dimension $k - i$.

PROPOSITION 2.11. *Let $\mathcal{A} = (A_1, \ldots, A_n)$ be an essential simple arrangement over the field $\mathbb{F}$ in $\mathbb{F}^k$. Let $z(\mathbf{x}) = \{j \in [n] : \mathbf{x} \in A_j\}$ and $r(\mathbf{x}) = r(A_{z(\mathbf{x})})$ the rank of $\mathbf{x}$ for $\mathbf{x} \in \mathbb{A}^k$. Then*

$$\mathcal{N}_i = \{\mathbf{x} \in \mathbb{A}^k : r(\mathbf{x}) \geq i\} \quad \textit{and} \quad \mathcal{M}_i = \{\mathbf{x} \in \mathbb{A}^k : r(\mathbf{x}) = i\}.$$

PROOF. See [**20**, Proposition 5.47]. □

PROPOSITION 2.12. *Let $\mathcal{A}$ be an essential simple arrangement over $\mathbb{F}_q$ and let $L = L(\mathcal{A})$ be the geometric lattice of $\mathcal{A}$. Then*

$$\mu_i(q^m) = |\mathcal{M}_i(\mathbb{F}_{q^m})|.$$

PROOF. See [**1**, Theorem 6.3] and [**20**, Proposition 5.48]. □

3. Coset leader and list weight enumerator

We repeat the definitions of the coset leader and list weight enumerators before we generalize them for extension codes.

DEFINITION 3.1. Let C be a linear code of length n over $\mathbb{F}_q$. Let $\mathbf{y} \in \mathbb{F}_q^n$. The *weight of the coset* $\mathbf{y}+C$ is defined by

$$\mathrm{wt}(\mathbf{y}+C) = \min\{\mathrm{wt}(\mathbf{y}+\mathbf{c}) : \mathbf{c} \in C\}.$$

A *coset leader* is a choice of an element $\mathbf{y} \in \mathbb{F}_q^n$ of minimal weight in its coset, that is, $\mathrm{wt}(\mathbf{y}) = \mathrm{wt}(\mathbf{y}+C)$. Let α_i be the number of cosets of C that are of weight i. Let λ_i be the number of vectors $\mathbf{y}$ in $\mathbb{F}_q^n$ that are of minimal weight i in its coset. Then $\alpha_C(X,Y)$, the *coset leader weight enumerator* of C, and $\lambda_C(X,Y)$, the *list weight enumerator* of C, are polynomials defined by

$$\alpha_C(X,Y) = \sum_{i=0}^{n} \alpha_i X^{n-i}Y^i \quad \text{and} \quad \lambda_C(X,Y) = \sum_{i=0}^{n} \lambda_i X^{n-i}Y^i$$

See [**14, 24**]. The *covering radius* $\rho(C)$ of C is the maximal i such that $\alpha_i(C) \neq 0$.

We have $\alpha_i = \lambda_i = \binom{n}{i}(q-1)^i$ for all $i \leq (d-1)/2$, where d is the minimum distance of C, since coset leaders are unique for these i. The coset leader weight enumerator gives a formula for the *probability of error*, that is the probability that the output of the decoder is the wrong codeword. In this decoding scheme the decoder uses the chosen coset leader as the error vector. See [**24**, §1.5]. The list weight enumerator is of interest in case the decoder has as output the list of all nearest codewords [**21, 30**].

Consider the functions $\alpha_i(T)$ and $\lambda_i(T)$ such that $\alpha_i(q^m)$ and $\lambda_i(q^m)$ are equal to the number of cosets of weight i and the number of elements in $\mathbb{F}_{q^m}^n$ of minimal weight i in its coset, respectively, with respect to the extended coded $C \otimes \mathbb{F}_{q^m}$.

DEFINITION 3.2. The *extended coset leader weight enumerator* and the *extended list weight enumerator* are defined by:

$$\alpha_C(X,Y,T) = \sum_{i=0}^{n} \alpha_i(T) X^{n-i}Y^i \quad \text{and} \quad \lambda_C(X,Y,T) = \sum_{i=0}^{n} \lambda_i(T) X^{n-i}Y^i.$$

In [**14**, Theorem 2.1] it is shown that the function $\alpha_i(T)$ is determined by finitely many data for all extensions of $\mathbb{F}_q$. This shows by Lagrange interpolation that the $\alpha_i(T)$ are polynomials in the variable T. In fact, let C be an $[n,k]$ code over $\mathbb{F}_q$. Then there are well defined nonnegative integers F_{ij} such that

$$\alpha_C(X,Y,T) = 1 + \sum_{i=1}^{n-k} \sum_{j=1}^{n-k} F_{ij}(T-1)(T-q)\cdots(T-q^{j-1})X^{n-i}Y^i$$

This is similar to the following expression of the extended weight enumerator in terms of the generalized weight enumerator, see [**14, 17, 18, 20, 22**].

$$A_w(T) = \sum_{r=0}^{k} \sum_{j=1}^{r} (T-1)(T-q)\cdots(T-q^{j-1})A_w^r.$$

REMARK 3.3. We have $\alpha_i(T) = \lambda_i(T) = \binom{n}{i}(T-1)^i$ for all $i \leq (d-1)/2$, where d is the minimum distance of C. An *information set* of C is a subset of size k of the positions of C where all possible q^k combinations of symbols occur. The columns of a generator matrix that correspond to an information set are linearly independent. Let $i(C)$ be the number of information sets of C. Then $\lambda_{n-k}(T) = i(C)\alpha_{n-k}(T)$.

REMARK 3.4. Note that the cosets of $\mathbf{y}$ and $\lambda\mathbf{y}$ have the same the weight for all $\lambda \in \mathbb{F}_q^*$. Hence $\alpha_i = (q-1)\bar{\alpha}_i$ and $\lambda_i = (q-1)\bar{\alpha}_i$ for all $i > 0$. Therefore the polynomials $\alpha_i(T)$ and $\lambda_i(T)$ are divisible by $T-1$ for all $i > 0$, that is there exist polynomials $\bar{\alpha}_i(T)$ and $\bar{\lambda}_i(T)$ such that $\alpha_i(T) = (T-1)\bar{\alpha}_i(T)$ and $\lambda_i(T) = (T-1)\bar{\lambda}_i(T)$ for all $i > 0$.

REMARK 3.5. Let d be the minimum distance of the code. If $i > (d-1)/2$ then there exists an error of weight i that is not uniquely correctable, so $\alpha_i < \lambda_i$. See [**14**]. The *Newton radius* is the maximal weight of a uniquely correctable error.

For some easy examples it is straightforward to determine the coset leader and list weight enumerators.

EXAMPLE 3.6. Let $C = \mathbb{F}_q^n$. Then $\lambda_C(X,Y,T) = \alpha_C(X,Y,T) = X^n$.

EXAMPLE 3.7. Let $C = \{\mathbf{0}\}$. Then $\lambda_i(T) = \alpha_i(T) = \binom{n}{i}(T-1)^i X^{n-i}Y^i$ and $\lambda_C(X,Y,T) = \alpha_C(X,Y,T) = (X + (T-1)Y)^n$.

EXAMPLE 3.8. Let C be the dual of an $[n,1,d]$ code. Then there are T cosets, corresponding to the elements of the ground field of the code. One is the code itself, the other $T-1$ cosets have each d elements of minimal weight 1. This gives $\lambda_C(X,Y,T) = X^n + d(T-1)X^{n-1}Y$ and $\alpha_C(X,Y,T) = X^n + (T-1)X^{n-1}Y$.

EXAMPLE 3.9. Let C be the $[n,1,n]$ repetition code. Then this code has not such an easy description of $\lambda_C(X,Y,T)$ and $\alpha_C(X,Y,T)$ as Example 3.8. Apart from the known expressions for $\lambda_i(T)$ and $\alpha_i(T)$ for i up to half the minimum distance $\lfloor (n-1)/2 \rfloor$ that hold for every code, we can determine α_{n-1} and λ_{n-1}. Let $\mathbf{a} = (a_1, \ldots, a_{i-1}, 0, a_{i+1}, \ldots, a_n)$ be a coset leader of weight $n-1$, then $a_1\mathbf{h}_1 + \ldots + a_{i-1}\mathbf{h}_{i-1} + a_{i+1}\mathbf{h}_{i+1} + \ldots + a_n\mathbf{h}_n = \mathbf{s}$. Since also $\mathbf{h}_1 + \ldots + \mathbf{h}_n = \mathbf{0}$, we can subtract a_j times that equation from the first one to obtain the coordinates of another coset leader. This coset leader also has to have weight $n-1$, because $\mathbf{a}$ is a coset leader. This means all coordinates of the coset leader have to be different, and every coset has n coset leaders. So we have that $\lambda_{n-1}(T) = n\alpha_{n-1}(T)$ and $\alpha_{n-1}(T) = (T-1)(T-2)\cdots(T-n+1)$. An alternative proof will be given in Example 5.5.

Although the extended weight enumerator contains a lot of information about a code, it does not determine the coset leader weight enumerator or even the covering radius of a code. See [**6**] for a counterexample. Also, all $[n,k,n-k+1]$ codes over $\mathbb{F}_q$ are called MDS (Maximum Distance Separable) and have the same generalized weight enumerator, but the covering radius varies for fixed n,k and q. See Example 5.30 of [**20**] and Example 4.10 in this paper.

DEFINITION 3.10. Let H be a parity check matrix of a linear $[n,k]$ code C over $\mathbb{F}_q$ and let $\mathbf{y}$ be a vector in $\mathbb{F}_q^n$. Let $\mathbf{s} = H\mathbf{y}^T$ be the *syndrome* of this word with respect to H. The *weight of* $\mathbf{s}$ *with respect to* H, also called the *syndrome weight* of $\mathbf{s}$, is defined by

$$\mathrm{wt}_H(\mathbf{s}) = \mathrm{wt}(\mathbf{y} + C).$$

Note that α_i is the number of syndromes in $\mathbb{F}_q^{n-k}$ with respect to H that are of weight i. See [**14**, Definition 2.1]. This number does not depend on the choice of H.

There is a one-to-one correspondence between cosets and syndromes. It is a well known fact that a coset leader corresponds to a minimal way to write its syndrome as a linear combination of the columns of a parity check matrix. The geometric interpretation of the weight of a coset and the syndrome weight is as follows. Let $\mathbf{h}_j$ be the j-th column of H and let $J \subseteq [n]$. Let V_J be the subspace of $\mathbb{F}_q^{n-k}$ that is generated by the vectors $\mathbf{h}_j$ with $j \in J$. Then we define the set

$$\mathcal{V}_t = \bigcup_{|J|=t} V_J.$$

PROPOSITION 3.11. *Let* $\mathbf{s}$ *in* $\mathbb{F}_q^{n-k}$ *be a syndrome with respect to* H*. Then*

$$\mathrm{wt}_H(\mathbf{s}) = t \quad \textit{if and only if} \quad \mathbf{s} \in \mathcal{V}_t \setminus \mathcal{V}_{t-1}.$$

COROLLARY 3.12. *Let* C *be a linear* $[n,k]$ *code with parity check matrix* H*. Then* α_t *is the number of vectors that are in the span of* t *columns of* H *but not in the span of* $t-1$ *columns of* H*.*

PROOF. Both the Proposition and the Corollary follow directly from the definitions of the syndrome weight and $\mathcal{V}_t$. □

REMARK 3.13. Note that Corollary 3.12 implies that $\alpha_i(T) = 0$ for all $i > n-k$, since $\mathcal{V}_{n-k}$ contains all vectors of $\mathbb{F}_q^{n-k}$. Furthermore

$$\sum_{i=0}^{n-k} \alpha_i = q^{n-k} \quad \text{and} \quad \sum_{i=0}^{n-k} \alpha_i(T) = T^{n-k}.$$

Hence

$$\sum_{i=1}^{n-k} \bar{\alpha_i} = \sum_{j=0}^{n-k-1} q^j \quad \text{and} \quad \sum_{i=1}^{n-k} \bar{\alpha_i}(T) = \sum_{j=0}^{n-k-1} T^j.$$

EXAMPLE 3.14. Let C be the binary Hamming code of length 7. Its parity check matrix consists of all possible nonzero vectors in $\mathbb{F}_2^3$, and the corresponding arrangement is shown in Figure 1. We can determine the extended coset leader weight enumerator by Corollary 3.12. As always, we have $\alpha_0(T) = 1$, this is the code itself. There are seven projective points in the arrangement, so $\bar{\alpha}_1(T) = 7$. On each of the seven lines there are $T+1$ points, of which we counted already three per line, so $\bar{\alpha}_2(T) = 7(T-2)$. Since $\bar{\alpha}_1(T) + \bar{\alpha}_2(T) + \bar{\alpha}_3(T) = T^2 + T + 1$, we find that $\bar{\alpha}_3(T) = (T-2)(T-4)$.
We see that $\rho(C) = 1$, $\rho(C \otimes \mathbb{F}_4) = 2$ and $\rho(C \otimes \mathbb{F}_{2^m}) = 3$ for $m \geq 3$. The list weight enumerator is equal to

$$\begin{aligned} \lambda_C(X,Y,T) \;=\; & X^7 + \\ & 7(T-1)X^6Y + \\ & 21(T-1)(T-2)X^5Y^2 + \\ & 28(T-1)(T-2)(T-4)X^4Y^3. \end{aligned}$$

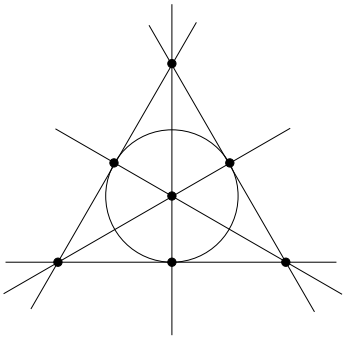

FIGURE 1. The hyperplane arrangement of the parity check matrix of the binary $[7, 4]$ Hamming code.

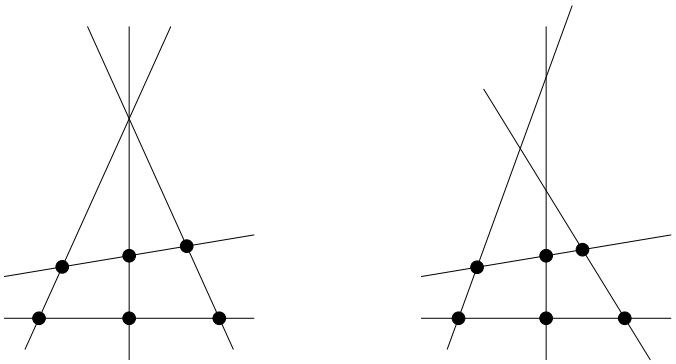

FIGURE 2. Two projective systems that induce the same lattice, but induce codes with different coset leader weight enumerators.

From the previous example, one might be tempted to think that the geometric lattice associated to the dual of a code is sufficient for determining the extended coset leader weight enumerator. But this is not true: we already mentioned the case of MDS codes. In Figure 2 we see two projective systems of six points. The points are on two lines in both cases, so the configurations give rise to equivalent geometric lattices. However, the corresponding dual codes do not have the same extended coset leader weight enumerator. In the left configuration, there is one syndrome that corresponds to a coset that has three different coset leaders of weight 2. In the right configuration, we do not have such a syndrome: instead, there are three cosets of weight 2, each with two coset leaders.

4. The derived code

To differentiate between codes with the same geometric lattice, we need the notion of a derived code and the corresponding derived arrangement: this is the arrangement with as hyperplanes all hyperplanes spanned by at least $k - 1$ points of the projective system $\mathcal{P}$. A similar notion of a derived configuration is given in [**10**, §5.10].

We will define the sets $\mathcal{V}_t$ again in a more formal way for an arbitrary $k \times n$ matrix G of rank k over a field $\mathbb{F}$ instead of a parity check matrix H of a code over $\mathbb{F}_q$.

DEFINITION 4.1. Let G be a $k \times n$ matrix of rank k over a field $\mathbb{F}$. Let $J \subseteq \{1, \ldots, n\}$. Let $\mathbf{g}_j$ be the j-th column of G. Let G_J be the $k \times |J|$ submatrix of G consisting of the columns $\mathbf{g}_j$ with j in J. Let $G_J(X)$ be the $k \times (|J|+1)$ matrix obtained by adding the column $X = (X_1, \ldots, X_k)^T$ to G_J, where $X_1, \ldots, X_k$ are variables. Let $G_{i,J}$ be the $(k-1) \times |J|$ matrix obtained by deleting the i-th row of G_J. Let $\Delta_J(X) = \det(G_J(X))$ and $\Delta_{i,J} = \det(G_{i,J})$ in case $|J| = k-1$. Let $D(G)$ be the *derived matrix* of G of size $k \times \binom{n}{k-1}$ with entries $\Delta_{i,J}$ with $i = 1, \ldots, k$ and $J \subseteq \{1, \ldots, n\}$ of size $k-1$ ordered lexicographically. Let $D_1(G)$ be the matrix obtained from $D(G)$ by deleting the zero columns. Let $D_2(G)$ the matrix obtained from $D_1(G)$ by deleting all columns that are a scalar multiple of a previous column.

Note that $D_2(G) = \bar{D}(G)$ is the *simplification* of $D(G)$.

REMARK 4.2. Suppose $|J| = k-1$. Then

$$\Delta_J(X) = \sum_{i=1}^{k} \Delta_{i,J} X_i. \quad \text{and} \quad \Delta_J(\mathbf{g}_j) = \sum_{i=1}^{k} \Delta_{i,J} g_{ij} = 0 \quad \text{for all } j \in J.$$

The columns of G_J are independent if and only if $\Delta_{i,J} \neq 0$ for some $i = 1, \ldots, k$. Moreover $\Delta_J(X)$ is the equation of the hyperplane that is generated by the columns of G_J if these are independent.

LEMMA 4.3. *Let G and G' be two $k \times n$ matrices of rank k over a field $\mathbb{F}$. If $G' = SG$ for some $k \times k$ invertible matrix S, then*

$$D(G') = \det(S) \cdot S^{-T} D(G),$$

where S^{-T} is the transpose of the inverse of S.

PROOF. Let $\Delta'_J(X) = \det(G'_J(X))$ and $\Delta'_{i,J} = \det(G'_{i,J})$ for $|J| = k-1$. Then

$$G'_J(X) = (G'_J | X) = (SG_J | X) = S(G_J | S^{-1}X).$$

It follows that

$$\begin{aligned} \sum_{j=1}^{k} \Delta'_{j,J} X_j &= \Delta'_J(X) \\ &= \det(S) \cdot \Delta_J(S^{-1}X) \\ &= \det(S) \cdot \sum_{j=1}^{k} \left(\sum_{i=1}^{k} \Delta_{i,J} (S^{-1})_{ij} \right) X_j. \end{aligned}$$

□

DEFINITION 4.4. Let G be a generator matrix of an $[n,k]$ code C over $\mathbb{F}_q$. The *derived codes* $D(C)$, $D_1(C)$ and $D_2(C)$ are defined by the generator matrices $D(G)$, $D_1(G)$ and $D_2(G)$, respectively.

REMARK 4.5. The definition of the derived codes $D(C)$, $D_1(C)$ and $D_2(C)$ does not depend on the chosen generator matrix by Lemma 4.3, since $\det(S) \cdot S^{-T}$ is an invertible matrix. Notice that $D(C)$ has length $\binom{n}{k-1}$. The code $D_1(C)$ has the same length as $D(C)$ if and only if every subset of $k-1$ columns of C are independent, that means $d(C^\perp) \geq k$ or, equivalently, $C^\perp$ is almost MDS.

EXAMPLE 4.6. Let $C = \mathbb{F}_q^n$ be the trivial code with generator matrix $G = I_n$. Then $D(G)$ has entries $(-1)^{i+j}\delta_{i,n+1-j}$. Hence $D(C) = C$.

EXAMPLE 4.7. Let C be a code of dimension 2 with generator matrix G. Then $D(G)$ has as in its first row $(-g_{21} \cdots - g_{2n})$ and in its second row $(g_{11} \cdots g_{1n})$. Hence $D(G)$ is also a generator matrix of C and $D(C) = C$.

EXAMPLE 4.8. Let C be the binary simplex code of dimension 3 and generator matrix

$$G = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 \end{pmatrix}.$$

Then $D(G) = D_1(G)$ is equal to

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 \end{pmatrix}$$

and $D_2(G)$ is equal to

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix}.$$

REMARK 4.9. In Example 4.8 we see that the binary simplex code C of dimension 3 is generalized equivalent with $D_2(C)$. This property holds for all q-ary simplex codes of any dimension. A converse is also true. Suppose that C is an $[n, k]$ code such that $k \geq 3$ and C and $D_2(C)$ are generalized equivalent codes. Consider the projective system $\mathcal{P}$ of the code. Assume moreover that on every line through two points of $\mathcal{P}$ there is a third point of $\mathcal{P}$. Then $\mathcal{P}$ defines a Desarguesian projective space of dimension $k-1$ that is embedded in the projective space of dimension $k-1$ over $\mathbb{F}_q$. Hence $\mathcal{P}$ is isomorphic with $\mathbb{P}^{k-1}(\mathbb{F})$, where $\mathbb{F}$ is a subfield of $\mathbb{F}_q$. See for example [**16**, §4.3].

EXAMPLE 4.10. Let G be the generator matrix of an MDS code with parameters $[n, k, n-k+1]$. Then all $k \times k$ submatrices of G have rank k, so all $k \times (k-1)$ submatrices of G have rank $k-1$. Hence $D(G)$ has no zero columns and $D(G) = D_1(G)$. We show that no two columns of $D(G)$ are multiples of each other. Suppose the contrary: there are two equal columns. Then there exist distinct subsets I and J of $[n]$ such that the hyperplanes A_I and A_J are the same. So the $\mathbf{g}_i$, $i \in I$ and $\mathbf{g}_j$, $j \in J$ all lie in the same hyperplane. Hence there exists a $j \in J \setminus I$ such that the k vectors $\mathbf{g}_i$, $i \in I \cup \{j\}$ lie in the same hyperplane. But that contradicts the fact that all $k \times k$ submatrices of G have rank k. Hence $D(G) = D_2(G)$ and the code $D(C)$ is simple and has length $\binom{n}{k-1}$. The converse is also true: $D_2(C)$ and $D(C)$ have the same length if and only if any k columns of C are independent, that means that C is MDS.
Now consider the arrangement $\mathcal{A} = \mathcal{A}_{D(G)}$ defined by $D(G)$. Let $\mathcal{P} = (P_1, \ldots, P_n)$ be the projective system defined by G. Consider P_j for some j. There are $\binom{n-1}{k-2}$ subsets I of $[n]$ of size $k-2$ such that $I \cup \{j\}$ has size $k-1$, hence there are $\binom{n-1}{k-2}$ hyperplanes of $\mathcal{A}$ going through P_j.
Let P be a point of the $k-1$ dimensional projective space that is distinct from P_j for all j. If P lies in l hyperplanes of $\mathcal{A}$, then there exist l subsets $I_1, \ldots, I_l$ of

$[n]$ such that $P \in A_{I_j}$ for all $j = 1, \ldots, l$. Then the I_j are mutually disjoint, since otherwise there are k points of $\mathcal{P}$ in a hyperplane. Hence P lies in at most $n/(k-1)$ hyperplanes. Then $\binom{n-1}{k-2}$ is the maximal number of hyperplanes of $\mathcal{A}$ going through a given point, since $\binom{n-1}{k-2} \geq n/(k-1)$. Therefore $D(C)$ has minimum distance $\binom{n}{k-1} - \binom{n-1}{k-2} = \binom{n-1}{k-1}$ by [**20**, Proposition 5.15].
If there exists an $[n, 3, n-2]$ code, then its derived code has parameters $[N, 3, d]$ with $N = \binom{n}{2}$ and $d = \binom{n-1}{2}$. Let d_m be the maximal value of the minimum distance of an $\mathbb{F}_q$-linear code of length N and dimension 3. In the following table we give the parameters q, n, N, d and d_m in the columns using [**7, 12, 13**]:

q	4			5			7					8						
n	6	5	4	6	5	4	8	7	6	5	4	10	9	8	7	6	5	4
N	15	10	6	15	10	6	28	21	15	10	6	45	36	28	21	15	10	6
d	10	**6**	**3**	10	6	3	21	15	10	6	3	36	28	21	15	10	6	3
d_m	11	**6**	**3**	11	7	4	23	17	12	7	4	38	30	24	17	12	8	4

q	9						
n	10	9	8	7	6	5	4
N	45	36	28	21	15	10	6
d	36	28	21	15	10	6	3
d_m	39	31	24	$17-18$	12	8	4

Only in two cases we obtain a code with maximum known d_m as a derived code.

EXAMPLE 4.11. This example is Example 2.28 from [**17**]. We consider two $[6, 3, 4]$ MDS codes C_1 and C_2 over $\mathbb{F}_9$, generated by the matrices

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & \alpha^5 & \alpha^6 \\ 0 & 0 & 1 & \alpha^3 & \alpha & \alpha^3 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & \alpha^7 & \alpha^4 & \alpha^6 \\ 0 & 0 & 1 & \alpha^5 & \alpha & 1 \end{pmatrix}$$

where $\alpha \in \mathbb{F}_9$ is the primitive element such that $\alpha^2 + \alpha - 1 = 0$. The derived codes $D(C_1)$ and $D(C_2)$ are generated by

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 1 & \alpha^4 & \alpha^6 & \alpha^3 & \alpha^7 & \alpha & 1 & \alpha^2 & 1 & \alpha^7 & 1 \\ 0 & 1 & 0 & \alpha^7 & 1 & 0 & 0 & \alpha^4 & 1 & 1 & 0 & \alpha^6 & \alpha & 1 & \alpha^3 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 0 & 0 & \alpha^7 & \alpha^2 & \alpha^3 & \alpha & 0 & \alpha^7 & \alpha^7 & \alpha^4 & \alpha^7 & \alpha & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & \alpha^3 & 0 & \alpha^6 & \alpha^6 & 0 & \alpha^7 & \alpha & \alpha^6 & \alpha^3 & \alpha \\ 0 & 0 & 1 & \alpha^5 & \alpha^5 & \alpha^6 & \alpha^3 & \alpha^7 & \alpha^4 & \alpha^3 & \alpha^5 & \alpha^2 & \alpha^4 & \alpha & \alpha^5 \end{pmatrix}.$$

The weight distributions of $D(C_1)$ and $D(C_2)$ are, respectively,

$$(1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 48, 0, 16, 312, 288, 64) \text{ and}$$
$$(1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 48, 0, 32, 264, 336, 48).$$

So the latter two codes are not generalized equivalent, and therefore not all $[6, 3, 4]$ MDS codes over $\mathbb{F}_9$ are generalized equivalent.

REMARK 4.12. It is not possible to define the notion of a derived matroid $D(M)$ for a given matroid M such that $D(M_C)$ is isomorphic with $M_{D(C)}$ for all codes C. The reason for this lies in the representation of matroids. It is possible to represent a matroid by various matrices. Even if those matrices are defined over the same field, it is possible that they generate nonequivalent codes. These nonequivalent

codes are likely to have different derived codes. This is the case in Example 4.11 where the two codes C_1 and C_2 have the same uniform matroid, but the matroids of the derived codes $D(C_1)$ and $D(C_2)$ are not isomorphic.

We can now give a more formal definition of the sets $\mathcal{V}_t$.

DEFINITION 4.13. Let G be a $k \times n$ matrix of rank k over a field $\mathbb{F}$. Let $J \subseteq \{1, \ldots, n\}$. Let V_J be the vector subspace of $\mathbb{F}^k$ that is generated by the vectors $\mathbf{g}_j^T$, $j \in J$. Then we define

$$\mathcal{V}_t = \bigcup_{|J|=t} V_J.$$

REMARK 4.14. Let L_G be the partial order with as elements the subspaces V_J and the inclusion as partial order. Then L_G is a geometric lattice, see Example 2.2. Let $\mathcal{A}_G$ be the arrangement of G. Let $L(\mathcal{A}_G)$ be the geometric lattice of the arrangement $\mathcal{A}_G$. Consider the map $\varphi : L_G \longrightarrow L(\mathcal{A}_G)$ defined by

$$\varphi(V_J) = A_J.$$

Recall from Definition 2.10 that $A_J = \cap_{j \in J} A_j$. Then φ is an isomorphism from L_G to $L(\mathcal{A}_G)$ as geometric lattices, see Example 2.3.

Consider the arrangement $\mathcal{A}_{D_2(G)}$. This arrangement is essential and simple. $\mathcal{A}_{D_2(G)}$ is the collection of all hyperplanes A_J with equation $\Delta_J(X) = 0$, where J consists of $k-1$ elements such that the $\mathbf{g}_j$, $j \in J$ are independent. Hence $A_J = V_J$ in this case.
For arbitrary J, if V_J has dimension l, then there is a $J' \subseteq J$ consisting of l elements such that $\mathbf{g}_i$, $i \in J'$ are independent. So $V_J = V_{J'}$. Now V_J is a subspace of the column space of G, which has dimension k. Hence there is an $I \subseteq [n]$ consisting of k elements such that $J' \subseteq I$ and the vectors $\mathbf{g}_i$, $i \in I$ are independent. So $V_{I \setminus \{i\}}$ is equal to the hyperplane $A_{I \setminus \{i\}}$ of the arrangement $\mathcal{A}_{D_2(G)}$ and

$$V_J = \bigcap_{i \in (I \setminus J')} A_{I \setminus \{i\}}.$$

is an intersection of the $k - l$ hyperplanes of the arrangement $\mathcal{A}_{D_2(G)}$.

DEFINITION 4.15. *Let G be a $k \times n$ matrix of rank k over a field $\mathbb{F}$. Let L_G be the geometric lattice of G. Let $L(\mathcal{A}_{D_2(G)})$ be the geometric lattice of the arrangement $\mathcal{A}_{D_2(G)}$. Define the map*

$$\psi : L_G \longrightarrow L(\mathcal{A}_{D_2(G)}) \quad \text{by} \quad \psi(V_J) = V_J.$$

REMARK 4.16. The map ψ is well defined , since V_J is an intersection of hyperplanes of the arrangement $\mathcal{A}_{D_2(G)}$. The map ψ is not a morphism of posets but an anti-morphism, since if $x \leq y$ in L_G, then $x \subseteq y$, hence $\psi(y) \leq \psi(x)$ in $L(\mathcal{A}_{D_2(G)})$. The map ψ is clearly injective, but not always surjective. Furthermore ψ is not an anti-morphism of lattices, that is the rules $\psi(x \wedge y) = \psi(x) \vee \psi(y)$ and $\psi(x \vee y)) = \psi(x) \wedge \psi(y)$ do not always hold. See Example 4.18.

REMARK 4.17. The matrix G gives rise to the decreasing sequence

$$\mathcal{N}_k \subset \mathcal{N}_{k-1} \subset \cdots \subset \mathcal{N}_1 \subset \mathcal{N}_0$$

of algebraic subsets of the affine space $\mathbb{A}^k$ as explained in Section 2. Furthermore we have the decreasing sequence

$$\mathcal{V}_0 \subset \mathcal{V}_1 \subset \cdots \subset \mathcal{V}_{k-1} \subset \mathcal{V}_k$$

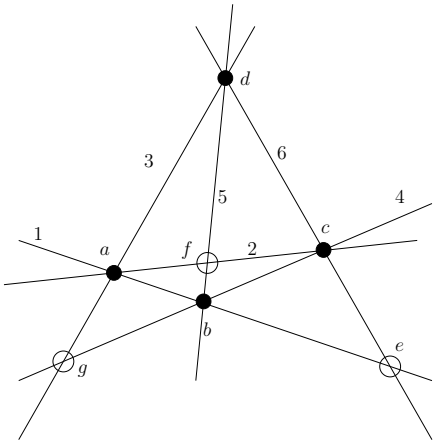

FIGURE 3. The projective system of G (the points a, b, c, d) and the arrangement of $D(G)$ (the lines $1, 2, 3, 4, 5, 6$).

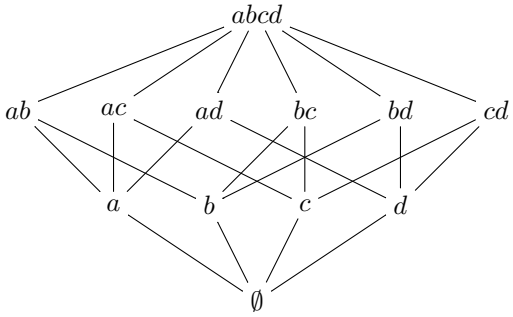

FIGURE 4. The Hasse diagram of the code in Example 4.18.

of the algebraic subsets $\mathcal{V}$ defined before. If $\mathbf{x} \in \mathcal{V}_t \setminus \mathcal{V}_{t+1}$, then $\mathbf{x} \in V_J$ for some J such that $t = |J| = \dim(V_J)$. Hence $\psi(V_J) = V_J \subseteq (\mathcal{N}_{k-t} \setminus \mathcal{N}_{k-t-1})$. Hence $(\mathcal{V}_t \setminus \mathcal{V}_{t+1}) \subseteq (\mathcal{N}_{k-t} \setminus \mathcal{N}_{k-t-1})$ and $\mathcal{V}_t \subseteq \mathcal{N}_{k-t}$ for all t. But it is not always the case that $\mathcal{V}_t = \mathcal{N}_{k-t}$, as we will see in Example 4.18.

EXAMPLE 4.18. Let G be the 3×4 matrix given by

$$G = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix}, \quad \text{then} \quad D(G) = \begin{pmatrix} 0 & 0 & 0 & 1 & -1 & 1 \\ 0 & -1 & 1 & 0 & 0 & -1 \\ 1 & 0 & -1 & 0 & 1 & 0 \end{pmatrix}.$$

A picture of the projective system is shown in Figure 3. The Hasse diagrams of the code and the derived code are given in Figures 4 and 5.

5. Coset leader weights and the derived code

We will now use the derived code, as defined in the previous section, to find a formal way to determine the extended coder leader weight enumerator and extended list weight enumerator.

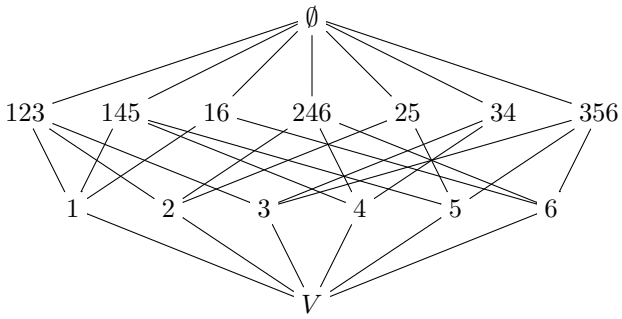

FIGURE 5. The Hasse diagram of the derived code in Example 4.18. On the second level, there are the points a, b, e, c, f, g, d respectively.

DEFINITION 5.1. Let H be a parity check matrix of the code C of an $[n, k]$ code. Let L_H be the geometric lattice of H. Let $\mathcal{A}$ be the arrangement of $D_2(H)$ and let $L = L(\mathcal{A})$ be the geometric lattice of $\mathcal{A}$ with rank function r. Let $\psi : L_H \to L$ be the map defined in Definition 4.15. Define for $x \in L$:

$$\hat{r}(x) = \max\{r(y) : y \in \psi(L_H), y \leq x\}.$$

REMARK 5.2. Note that $\hat{r}(x) = r(x)$ for $x \in L$ if and only if $x \in \psi(L_H)$. Hence $r(x) = \hat{r}(x)$ for all $x \in L$ if and only if $L = \psi(L_H)$.

We can now determine the extended coset leader weight enumerator. The formula for $\alpha_i(T)$ is analogous to the one for $\mu_i(T)$ of Proposition 2.12.

THEOREM 5.3. *The coefficients of the extended coset leader weight enumerator are given by*

$$\alpha_i(T) = \sum_{\substack{x \in L \\ \hat{r}(x) = n-k-i}} \chi_{L_x}(T).$$

PROOF. Suppose we have a syndrome vector $\mathbf{x} \in \mathbb{A}^{n-k}$. We want to determine its syndrome weight $\mathrm{wt}_H(\mathbf{x})$. We do this by looking at the elements of L, which are subspaces of $\mathbb{A}^{n-k}$. Let $x \in L$ be the element in L with the highest rank that contains $\mathbf{x}$. Such an element exists, because every syndrome vector is contained in $0_L = \mathbb{A}^{n-k}$ and if a vector is contained in some $y \in L$, then it is also contained in all elements covered by y. Furthermore, this element x is unique, since if there were more, $\mathbf{x}$ would be in there intersection, which is an element of L of higher rank. In fact, $\mathbf{x}$ is contained in all elements of the sublattice L^x.
We distinguish between the two cases $x \in \psi(L_H)$ and $x \notin \psi(L_H)$. First, assume $x \in \psi(L_H)$ and $\dim x = t$. Then $r(x) = n - k - t$ and also $\hat{r}(x) = n - k - t$. Now $\mathbf{x}$ can be written as a linear combination of t vectors of H: the columns corresponding to the coatoms bigger than x in L, or, equivalently, the atoms smaller than x in L_H. But since x has maximal rank such that $\mathbf{x} \in x$, we have that $\mathbf{x}$ can not be written as a linear combination of $t - 1$ columns of H. This means $\mathbf{x} \in \mathcal{V}_t \setminus \mathcal{V}_{t-1}$ and thus $\mathrm{wt}_H(\mathbf{x}) = t$.
Now suppose $x \notin \psi(L_H)$. Then there is a $y \in \psi(L_H)$ such that $y \leq x$ and there is

no element of higher rank with that property. This y is not necessarily unique, but its rank is: it is equal to $\hat{r}(x)$. Let $\hat{r}(x) = n-k-t = r(y)$, then y is of dimension t and $\mathbf{x} \in \mathcal{V}_t \setminus \mathcal{V}_{t-1}$ because the rank of y is maximal.
Combining this, we see that for all vectors $\mathbf{x} \in x$ that are not in any element bigger than x, we have $\mathrm{wt}_H(\mathbf{x}) = t$. The number of such vectors is given by $\chi_{L_x}(q^m)$, as was explained in Remark 2.9. This proves the given formula. □

REMARK 5.4. If $L = \psi(L_H)$, then $\alpha_i(T) = \mu^*_{n-k-i}(T)$, where $\mu^*_i(T)$ is the Möbius polynomial of the dual code.

EXAMPLE 5.5. Consider the $[n, 1, n]$ repetition code with parity check matrix $H = (I_{n-1}|\mathbf{h}_n)$ with $\mathbf{h}_n = (-1, \ldots, -1)^T$. Let A_{ij} be the hyperplane spanned by all column vectors of H except $\mathbf{h}_i$ and $\mathbf{h}_j$. Then A_{ij} is given by the equation $X_i - X_j = 0$ if $1 \leq i < j < n$ and by $X_i = 0$ if $1 \leq i < j = n$. The arrangement of $\mathcal{A}_{D(H)}$ is simple and consists of the A_{ij}.
The geometric lattice $L(\mathcal{A}_{D(H)})$ has the following alternative combinatorial description by collections P of subsets of $\{1, \ldots, n\}$. The minimum element and the maximum element both consist of one subset, the empty set and the whole set, respectively. Otherwise P is a collection of mutual disjoint subsets of $\{1, \ldots, n\}$ with at least 2 elements. Then this P corresponds one-to-one to the intersection of all hyperplanes A_{ij} such that there exists a subset $I \in P$ such that $i, j \in I$. If P is not the minimum element, then the rank of P is equal to $|\cup P| - |P|$, that is if P consists of the mutual disjoint subsets $I_1, \ldots, I_t$, then

$$r(P) = \sum_{i=1}^{t} |I_t| - t.$$

The partial order is given by $P \leq P'$ if and only if for all $I \in P$ there exists (a unique) $I' \in P'$ such that $I \subseteq I'$.
The vector subspace V_J of $\mathbb{F}^{n-1}$ is generated by the vectors $\mathbf{h}_j^T$, $j \in J$. If $|J| = n-1$, then $V_J = \mathbb{F}^{n-1}$. Now $L(H)$ consists of all V_J with J a subset of $\{1, \ldots, n\}$ such that $|J| \neq n-1$. The P that corresponds to $\psi(V_J)$ is the singleton consisting of the unique subset $J^c = \{1, \ldots, n\} \setminus J$.
Let P consist of the mutual disjoint subsets $I_1, \ldots, I_t$. Let Q corresponds to an element $\psi(V_J)$. Then $Q \leq P$ if and only if here exists an i such that $J^c \subseteq I_i$. Hence

$$\hat{r}(P) = \max_{i=1,\ldots,t} |I_i| - 1$$

if P is not the minimum element and $\hat{r}(P) = 0$ if and only if P is the minimum element. So $\alpha_{n-1}(T) = \chi_{L(\mathcal{A}_{D(H)})}(T) = (T-1)(T-2)\cdots(T-n+1)$ by [**28**, Corollary 2.2] and [**1, 29**]. This is in accordance with Example 3.9

For the determination of the extended list weight enumerator, we can take a similar approach. We now need to calculate not only the coset weight of a syndrome vector, but also the number of different coset leaders. We do this with the following multiplicity function, defined for all $x \in L$:

$$m(x) = |\{y \in \psi(L_H) : y \leq x, r(y) = \hat{r}(x)\}|.$$

REMARK 5.6. The number $m(x)$ counts the number of elements of maximal rank in L such that they are below x and elements of $\psi(L_H)$. By the definition of $\hat{r}$, we have that $m(x) = 1$ if $r(x) = \hat{r}(x)$.

The formula for $\lambda_i(T)$ is now as follows.

THEOREM 5.7.

$$\lambda_i(T) = \sum_{\substack{x \in L \\ \hat{r}(x)=n-k-i}} m(x) \cdot \chi_{L_x}(T).$$

PROOF. Let $\mathbf{x}$ and x be defined as in the proof of Theorem 5.3. Again, we distinguish between the two cases $x \in \psi(L_H)$ and $x \notin \psi(L_H)$.
If $x \in \psi(L_H)$, then from the proof of Theorem 5.3 we know that $\mathbf{x}$ is a linear combination of the atoms smaller than x in L_H. Because L_H is an atomic lattice, this linear combination is unique. So $\mathbf{x}$ is a syndrome that corresponds to a coset with a unique coset leader. For these cosets, their contribution to the extended list weight enumerator is the same as their contribution to the coset leader weight enumerator. This means we should have $m(x) = 1$, which is indeed the case as we noticed in Remark 5.6.
Now let $x \notin \psi(L_H)$. From the proof of Theorem 5.3 we know there is a $y \in \psi(L_H)$ such that $y \leq x$ and there is no element of higher rank with that property. As noticed, this y does not need to be unique. In fact, the number of y's with this property is exactly $m(x)$. Every y is a unique join of elements of L_H, so every y corresponds to a way of writing $\mathbf{x}$ as a linear combination of columns of H. Therefore, the contribution of $\mathbf{x}$ to the extended list weight enumerator is $m(x)$ times its contribution to the extended coset leader weight enumerator. □

We illustrate the previous theorems by several examples.

EXAMPLE 5.8. Let C be an $[n, n-2, 3]$ code. Let H be a parity check matrix of C. Then $D(H)$ and H generate the same code $C^\perp$ by Example 4.7. Furthermore $C^\perp$ is an MDS code. Hence $D_2(H) = D(H)$ and the extended coset leader weight enumerator of C is equal to the Möbius polynomial of $C^\perp$:

$$\alpha_1(T) = \mu_1(T) = n(T-1) \quad \text{and} \quad \alpha_2(T) = \mu_2(T) = (T-1)(T-n).$$

EXAMPLE 5.9. Let C be the q-ary Hamming code of redundancy r and length $(q^r-1)/(q-1)$. Then $C^\perp$ and $D_2(C^\perp)$ are both general equivalent to the simplex code, see Example 4.8. Hence by Remark 5.4 the extended coset leader weight enumerator of C is equal to the Möbius polynomial of the simplex code, which is computed in [**20**, Example 5.53]:

$$\alpha_i(T) = \mu_{r-i}(T) = \begin{bmatrix} r \\ i \end{bmatrix}_q (T-1)(T-q)\cdots(T-q^{i-1}),$$

where $\begin{bmatrix} r \\ i \end{bmatrix}_q$ is the Gaussian binomial.

EXAMPLE 5.10. Let C be an MDS code with parameters $[n, n-3, 4]$ with parity check matrix H over the field $\mathbb{F}_q$. Then the projective system $\mathcal{P}$ of H consists of n points in the projective plane $\mathbb{P}^2(\mathbb{F}_q)$ such that no three points of $\mathcal{P}$ are on a line. This means that $\mathcal{P}$ is an *arc*. See for example Chapter 8 of [**16**]. Lines that intersect $\mathcal{P}$ in 0, 1 or 2 points are called exterior lines, tangents and secants, respectively.
Suppose q is odd and $\mathcal{P}$ is a complete arc. Then $n = q+1$ and $\mathcal{P}$ is a conic. It is a classical result that the points and lines of $\mathbb{P}^2(\mathbb{F}_q)$ can be divided into three types. See for example §8.1 of [**16**]. For the points, it is as follows:

- There are $\binom{q+1}{2}$ external points of $\mathcal{C}$. Through such a point are two tangents of $\mathcal{C}$, $\frac{1}{2}(q-1)$ secants of $\mathcal{C}$, and $\frac{1}{2}(q-1)$ exterior lines of $\mathcal{C}$.

- There are $q+1$ points on $\mathcal{C}$. Through such a point there is one tangent of $\mathcal{C}$ and q secants of $\mathcal{C}$.
- There are $\binom{q}{2}$ internal points of $\mathcal{C}$. Through such a point are no tangents of $\mathcal{C}$, $\frac{1}{2}(q+1)$ secants of $\mathcal{C}$, and $\frac{1}{2}(q+1)$ exterior lines of $\mathcal{C}$.

Dually, we have the following types of lines:

- There are $\binom{q+1}{2}$ secants of $\mathcal{C}$. On such a line, two points are on $\mathcal{C}$, $\frac{1}{2}(q-1)$ points are internal points of $\mathcal{C}$ and $\frac{1}{2}(q-1)$ points are external points of $\mathcal{C}$.
- There are $q+1$ tangents to $\mathcal{C}$.
- There are $\binom{q}{2}$ exterior lines of $\mathcal{C}$. On such a line, $\frac{1}{2}(q+1)$ points are internal points of $\mathcal{C}$ and $\frac{1}{2}(q+1)$ points are external points of $\mathcal{C}$.

The derived arrangement $\mathcal{A}$ of H consists of the $\binom{q+1}{2}$ secants of $\mathcal{C}$. All points of $\mathbb{P}^2(\mathbb{F}_q)$ are on at least two lines of $\mathcal{A}$, i.e., are intersections of the lines of $\mathcal{A}$, if $q>3$. Assume $q>3$. The lattice $L=L(\mathcal{A})$ has on the first level L_1 all lines of $\mathcal{A}$ and $\hat{r}(x)=r(x)=1$ if x is such a line. For such a line we have $\chi_{L_x}(T)=T^2-(q+1)T+q=(T-1)(T-q)$ by Definition 2.4.
The second level L_2 consists of all the points of the plane $\mathbb{P}^2(\mathbb{F}_q)$. For these points, $\hat{r}(x)=2$ if x is a point of $\mathcal{P}$ and $\hat{r}(x)=1$ otherwise. For all these points we have $\chi_{L_x}(T)=T-1$ by Definition 2.4.
As usual, $\alpha_0(T)=1$. For $\alpha_1(T)$, we look at all x with $\hat{r}(x)=2$. These are exactly the points of $\mathcal{P}$ and there are $q+1$ of them, so

$$\alpha_1(T)=\sum_{\hat{r}(x)=2}\chi_{L_x}(T)=(q+1)(T-1).$$

For $\alpha_2(T)$, we look at all x with $\hat{r}(x)=1$. These are the points not in $\mathcal{P}$ and the lines of $\mathcal{A}$, so

$$\begin{aligned}\alpha_2(T) &= \sum_{\hat{r}(x)=1}\chi_{L_x}(T)\\ &= (q^2+q+1-(q+1))(T-1)+\tbinom{q+1}{2}(T-1)(T-q)\\ &= (T-1)(\tbinom{q+1}{2}T+q^2-q\tbinom{q+1}{2}).\end{aligned}$$

We can determine $\alpha_3(T)$ by Remark 3.13:

$$\begin{aligned}\alpha_3(T) &= (T-1)(T^2+T+1-(q+1)-(\tbinom{q+1}{2}T+q^2-q\tbinom{q+1}{2}))\\ &= (T-1)(T^2+(1-\tbinom{q+1}{2})T-q(q+1)+q\tbinom{q+1}{2})\end{aligned}$$

In case $q=3$ we get similarly:

$$\begin{aligned}\alpha_0(T) &= 1\\ \alpha_1(T) &= 4(T-1)\\ \alpha_2(T) &= 3(T-1)+6(T-1)(T-2)\\ &= (T-1)(6T-9)\\ \alpha_3(T) &= (T^2+T+1-(6T-9))(T-1)\\ &= (T^2-5T+10)(T-1)\end{aligned}$$

We will now determine the extended list weight enumerator for $q>3$. Again, $\lambda_0(T)=1$. For $\lambda_1(T)$, note that $m(x)=1$ for all x with $\hat{r}(x)=2$ because of

Remark 5.6. So

$$\lambda_1(T) = \alpha_1(T) = (q+1)(T-1).$$

For $\lambda_2(T)$, look at the x such that $\hat{r}(x) = 1$. Then x is a line of $\mathcal{A}$ or a point not on $\mathcal{P}$. In the first case, $m(x) = 1$. In the second case, we have to count how many lines of $\mathcal{A}$ go through the point x. This depends on x being an internal or external point of $\mathcal{P}$.

$$\begin{aligned} \bar{\lambda}_2(T) &= \tbinom{q+1}{2}(T-q) + \tbinom{q+1}{2} \cdot \tfrac{1}{2}(q-1) + \tbinom{q}{2} \cdot \tfrac{1}{2}(q+1) \\ &= \tbinom{q+1}{2}(T-1). \end{aligned}$$

The code is MDS. So every triple is an information set and the number of information sets of C is $i(C) = \binom{q+1}{3}$. Furthermore $\lambda_3(T) = i(C)\alpha_3(T)$ by Remark 3.3. Hence

$$\begin{aligned} \bar{\lambda}_3(T) &= \tbinom{q+1}{3}\left(T^2 + T + 1 - (q^2 + q + 1 + \tbinom{q+1}{2}(T-q)\right) \\ &= \tbinom{q+1}{3}(T-q)(T - \tfrac{1}{2}(q-2)(q+1)). \end{aligned}$$

Example 5.11. Let C be an MDS code with parameters $[n, n-3, 4]$ with parity check matrix H over the field $\mathbb{F}_q$. We saw in Example 5.10 that the points of $\mathcal{P}$ form an arc in the projective plane. An arc in $\mathbb{P}^2(\mathbb{F}_q)$ is complete if and only if every point of $\mathbb{P}^2(\mathbb{F}_q)$ lies on a secant of the arc. If q is even and $n > q - \sqrt{q} + 1$, then $\mathcal{P}$ is contained in a complete arc of $q+2$ elements. If moreover $(n,q) \neq (2,2)$ then this complete arc is unique. See [**16**, Corollary 2 of Theorem 10.3.3]. In case q is odd and $n > q - \frac{1}{4}\sqrt{q} + \frac{7}{4}$, then $\mathcal{P}$ is contained in a unique complete arc of $q+1$ elements, that is a conic. See [**16**, Theorem 10.4.4] and for recent improvements: "Planar arcs" by S. Ball and M. Lavrauw, arXiv:1705.10940v4.
Assume that q is odd and $n > q - \frac{1}{4}\sqrt{q} + \frac{7}{4}$. Then $\mathcal{P}$ is contained in a unique conic $\mathcal{C}$ of $q+1$ elements. A point Q in the projective plane over $\mathbb{F}_q$ lies on a secant of $\mathcal{P}$ if and only if $Q \notin \mathcal{C} \setminus \mathcal{P}$.
The points of $\mathcal{P}$ correspond one-to-one to the projective syndromes of weight 1. So $\bar{\alpha}_1(T) = n$.
All points outside $\mathcal{C}$ correspond one-to-one to projective syndromes of weight 2 over $\mathbb{F}_q$. The projective syndromes of weight 2 over $\mathbb{F}_{q^m}$ but not over $\mathbb{F}_q$ correspond one-to-one to a point on a unique line between two distinct points of $\mathcal{P}$, since a point that lies on two secant lines of $\mathcal{P}$ is defined over $\mathbb{F}_q$. Hence

$$\begin{aligned} \bar{\alpha}_2(T) &= q^2 + q + 1 - (q+1) + \tbinom{n}{2}(T + 1 - (q+1)) \\ &= \tbinom{n}{2}T + q^2 - q\tbinom{n}{2}. \end{aligned}$$

Finally $\bar{\alpha}_1(T) + \bar{\alpha}_2(T) + \bar{\alpha}_3(T) = T^2 + T + 1$ by 3.13. Therefore

$$\bar{\alpha}_3(T) = T^2 + (1 - \tbinom{n}{2})T + q\tbinom{n}{2} - q^2 - n + 1.$$

Note that when substituting $n = q+1$, we get the formulas from Example 5.10. If q even and $n > \frac{1}{2}(q+2)$, then similarly

$$\begin{aligned} \bar{\alpha}_2(T) &= \tbinom{n}{2}T + q^2 - 1 - q\tbinom{n}{2}, \\ \bar{\alpha}_3(T) &= T^2 + (1 - \tbinom{n}{2})T + q\tbinom{n}{2} - q^2 - n + 2. \end{aligned}$$

The derived arrangement is more difficult from the case in Example 5.10. Therefore determining the list weight enumerator is also not easy.

6. MacWilliams type property for duality

Research Problem 5.1 in [**24**, Chapter 5] asked whether the coset leader weight enumerator of C determines the coset leader weight enumerator of $C^\perp$, as is the case for the ordinary weight enumerator by the MacWilliams relations. This problem has a negative answer by [**2**]. The authors give three binary [15,3,7] codes that have the same coset leader weight enumerator, but the dual codes have mutually distinct coset leader weight enumerators. In fact a much smaller counterexample will do.

EXAMPLE 6.1. The two codes of length 3 with parity check matrices $H_1 = (110)$ and $H_2 = (111)$ both have the same extended coset leader weight enumerator $X^3 + (T-1)X^2Y$. But their dual codes have distinct extended coset leader weight enumerator, since

$$\alpha_{C_1^\perp}(X,Y,T) = X^3 + 2(T-1)X^2Y + (T-1)XY^2$$

$$\alpha_{C_2^\perp}(X,Y,T) = X^3 + 3(T-1)X^2Y + (T-1)(T-2)XY^2.$$

Remark that the code $C_1^\perp$ is degenerate. A non degenerate counterexample is obtained as follows.

EXAMPLE 6.2. Let C_3 and C_4 be the two $[6,3]$ codes over $\mathbb{F}_2$ with generator matrices

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{pmatrix}.$$

The next table shows the coefficients of the extended coset leader weight enumerator and the extended list weight enumerator of the codes and their duals. The values for $i = 0$ are left out, since they are all equal to 1 because of the zero word.

	i	C_3	C_4
$\alpha_{C,i}$	1	$5(T-1)$	$5(T-1)$
	2	$2(T-1)(3T-5)$	$2(T-1)(3T-5)$
	3	$(T-1)(T-2)(T-3)$	$(T-1)(T-2)(T-3)$
$\alpha_{C^\perp,i}$	1	$4(T-1)$	$5(T-1)$
	2	$3(T-1)(2T-3)$	$2(T-1)(3T-5)$
	3	$(T-1)(T-2)(T-3)$	$(T-1)(T-2)(T-3)$
$\lambda_{C,i}$	1	$6(T-1)$	$6(T-1)$
	2	$2(T-1)(7T-12)$	$2(T-1)(7T-11)$
	3	$12(T-1)(T-2)(T-3)$	$13(T-1)(T-2)(T-3)$
$\lambda_{C^\perp,i}$	1	$6(T-1)$	$6(T-1)$
	2	$13(T-1)^2$	$2(T-1)(7T-11)$
	3	$12(T-1)(T-2)(T-3)$	$13(T-1)(T-2)(T-3)$

We see that the extended coset leader weight enumerator of the two codes are equal, but none of the other polynomials, so they are not defined by the extended coset leader weight enumerator.

7. Directions for further work

We have given a formal way to determine the extended coset leader and list weight enumerator, via the derived code. Several examples were considered. We also studied to which extend the several polynomials define each other. The theory and examples give rise to new research problems.

The structure of the derived arrangement is actually studying dependencies between dependencies of points in the projective space. In computational geometry, these is knows as *second order syzygies*. See [**10**] for an introduction to the topic. For determining the coset leader and list weight enumerator of a given code, the language of computational geometry could be suitable.

It was mentioned in Remark 4.12 that the derived code is not a matroid invariant. However, one could define a 'derived matroid' as the most general derived arrangement possible. For example, in Figure 2, the situation on the left should never happen: three lines in the derived arrangement should be in the most general possible position, so they can not intersect in a point. Studying this object might give us more information between invariants of a code that are matroid invariants, such as the extended weight enumerator, and the extended coset leader weight enumerator.

In Examples 5.10 we considered a Reed-Solomon code of codimension 3. Can we generalize the methods in this example to other dimensions? Example 5.5 calculates $\alpha_i(T)$ and $\lambda_i(T)$ for some cases, but more calculations are necessary to complete the determination of the extended coset leader and list weight enumerator. Are all α_i a product of linear factors?

If we repeat the procedure of taking the derived code, we will eventually end up with the simplex code, i.e., all hyperplanes of the projective space belong to the derived code. We wonder how fast this process converges, and what it tells us about the code.

In Section 6 we showed by counterexample that the extended coset leader weight enumerator is not determined by the same polynomial from the dual code. It seems unlikely that such a relation exists for the extended list weight enumerator, but so far there is no counterexample known.

References

1. C. A. Athanasiadis, *Characteristic polynomials of subspace arrangements and finite fields*, Adv. Math. **122** (1996), 193–233.
2. T. Baicheva, I. Bouyukliev, S. Dodunekov, and W. Willems, *Teaching linear codes*, International Congress MASSEE, 2003.
3. A. Barg, *Complexity issues in coding theory*, Handbook of coding theory (V. S Pless, W.C. Huffman, and R. Brualdi, eds.), vol. 1, North-Holland, Amsterdam, 1998, pp. 649–754.
4. E. R. Berlekamp, R. J. McEliece, and H. C. A. van Tilborg, *On the inherent intractability of certain coding problems*, IEEE Trans. Inform. Theory **24** (1978), 384–386.
5. A. Björner and T. Ekedahl, *Subarrangments over finite fields: Chomological and enumerative aspects*, Adv. Math. **129** (1997), 159–187.
6. T. Britz and C. G. Rutherford, *Covering radii are not matroid invariants*, Discrete Math. **296** (2005), 117–120.

7. A.E. Brouwer, *Bounds on the size of linear codes*, Handbook of Coding Theory (Vera S. Pless and W.Cary Huffman, eds.), Elsevier, Amsterdam, 1998, pp. 295–461.
8. T. H. Brylawski and J. G. Oxley, *The Tutte polynomial and its applications*, Matroid Applications (N. White, ed.), Cambridge University Press, Cambridge, 1992, pp. 173–226.
9. H. Crapo, *The Tutte polynomial*, Aequationes Math. **3** (1969), 211–229.
10. ———, *An algebra of pieces of space – Hermann Grassmann to Gian Carlo Rota*, From Combinatorics to Philosophy (E. Damiani, O. D'Antona, V. Marra, and F. Palombi, eds.), Springer, New York, 2009, pp. 61–90.
11. H. Crapo and G. C. Rota, *On the foundations of combinatorial theory: Combinatorial geometries*, MIT Press, Cambridge MA, 1970.
12. M. Grassl, *Searching for linear codes with large minimum distance*, Discovering Mathematics with Magma — Reducing the Abstract to the Concrete (Wieb Bosma and John Cannon, eds.), Algorithms and Computation in Mathematics, vol. 19, Springer, Heidelberg, 2006, pp. 287–313.
13. ———, *Bounds on the minimum distance of linear codes and quantum codes*, Online available at `http://www.codetables.de`, 2007.
14. T. Helleseth, *The weight distribution of the coset leaders of some classes of codes with related parity-check matrices*, Discrete Math. **28** (1979), 161–171.
15. T. Helleseth and T. Kløve, *The Newton radius of codes*, IEEE Trans. Inform. Theory **43** (1997), no. 6, 1820–1831.
16. J. W. P. Hirschfeld, *Projective geometries over finite fields*, second ed., Oxford Mathematical Monographs, The Clarendon Press Oxford University Press, New York, 1998.
17. R. P. M. J. Jurrius, *Codes, arrangements, matroids, and their polynomial links*, Ph.D. thesis, Eindhoven University of Technology, 2012.
18. R. P. M. J. Jurrius and R. Pellikaan, *Extended and generalized weight enumerators*, Proc. Int. Workshop on Coding and Cryptography WCC-2009 (T. Helleseth and Ø. Ytrehus, eds.), Selmer Center, Bergen, 2009, pp. 76–91.
19. ———, *The extended coset leader weight enumerator*, Proc. 30th Symposium 2009 on Information Theory in the Benelux (F. Willems and T. Tjalkens, eds.), 2009, pp. 217–224.
20. ———, *Codes, arrangements and matroids*, Algebraic Geometry Modeling in Information Theory (Edgar Martínez-Moro, ed.), vol. 8, World Scientific, New Jersey, 2013, pp. 219–325.
21. J. Justesen and T. Høholdt, *Bounds on list decoding of MDS codes*, IEEE Trans. Inform. Theory **47** (2001), 1604–1609.
22. T. Kløve, *Support weight distribution of linear codes*, Discrete Math. **106/107** (1992), 311–316.
23. ———, *Codes for error detection*, Series on Coding Theory and Cryptology, vol. 2, World Scientific Publishing, 2007.
24. F. J. MacWilliams and N. J. A. Sloane, *The theory of error-correcting codes*, North-Holland Mathematical Library, Amsterdam, 1977.
25. M. Munuera, *Steganography and error-correcting codes*, Signal Processing **87** (2007), no. 6, 1528–1533.
26. ———, *Steganography from a coding theory point of view*, Algebraic Geometry Modeling in Information Theory (Edgar Martínez-Moro, ed.), vol. 8, World Scientific, New Jersey, 2013, pp. 83–128.
27. P. Orlik and H. Terao, *Arrangements of hyperplanes*, vol. 300, Springer-Verlag, Berlin, 1992.
28. R. P. Stanley, *Enumerative combinatorics*, vol. 1, Cambridge University Press, Cambridge, 1997.
29. R. P. Stanley, *An introduction to hyperplane arrangements*, Geometric combinatorics, IAS/Park City Math. Ser., no. 13, Amer. Math. Soc., Providence, RI, 2007, pp. 389–496.
30. M. Sudan, *Decoding of Reed-Solomon codes beyond the error-correction bound*, J. Complexity **13** (1997), 180–193.
31. A. Vardy, *The intractability of computing the minimum distance of a code*, IEEE Trans. Inform. Theory **43** (1997), 1757–1766.
32. T. Zaslavsky, *Facing up to arrangements: Face-count formulas for partitions of space by hyperplanes*, vol. 1, Amer. Math. Soc., Providence, RI, 1975.
33. ———, *Signed graph colouring*, Discrete Math. **39** (1982), 215–228.

Relinde Jurrius, Vrije Universiteit Brussel, Belgium
E-mail address: Relinde.Jurrius@vub.ac.be

Ruud Pellikaan, Eindhoven University of Technology, The Netherlands
E-mail address: G.R.Pellikaan@tue.nl